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Relationship Between Weyl's Curvature Tensor and Other Curvature Tensors

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Abstract
In this paper, the authors investigated the relation between the Weyl's projective curvature tensor and some tensors, where the tensor W^i_{jkh} satisfies a generalized four recurrent property with respect to Cartan's covariant derivative. We call this type of space a generalized W^h-four recurrent Finsler space and denote by GW^h-FRF_n .

List of Symbols

W^i_{jkh}: *Weyl's Projective Curvature Tensor ;*
R^i_{jkh} : *Cartan's Third Curvature Tensor ;*
: *Cartan's second curvature tensor* ;P^i_{jkh}
: *Cartan's forth curvature tensor* ;K^i_{jkh}
H^i_{jkh} : *Berwald Curvature Tensor ;*
W_{kh} : *Ricci Tensor ;*
W_k : *Curvature Vector*
W_{jrkh} : *associate curvature tensor for* W^i_{jkh}

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1. Introduction

Rund [27] introduced and studied A3-dimensional Riemannian space of recurrent. The generalized curvature tensors in recurrent Finsler space for first, second and third order by using the sense of Cartan's covariant derivative was discussed by ([3], [8], [11], [12], [15], [22], [23] and [24]). Some properties for Weyl's projective curvature tensor were studied by AL-Qashbari [7], Ahsan and Ali [1], Abu-Donia, Shenawy, and Abdehameed, [2], Emamian and Tayebi [15] and Qasem and Saleem [26]. The generalized birecurrent, trirecurrent Finsler space and higher order recurrent were studied in ([5], [6], [14], [17], [18], [20] and [25]). Also, Ahsan and Ali [4] introduced the curvature tensor for the spacetime of general relativity. Complete Finsler spaces of constant negative Ricci curvature was studied by Bidabad and Sepasi [13]. Qasem, and Saleem, [26] studied on W^h_{jkh} generalized birecurrent Finsler space, W^h-recurrent space. Decomposability of projective curvature tensor in recurrent Finsler space was studied by AL-Qashbari [9], Al-Qufail [10] and Thakur, Mishra, Lodhi [28] and others.

Let the components of the corresponding metric tensor and Berwald's connection coefficients be denoted by g_{ij} and G^i_{jk} respectively. These are positively homogeneous of degree zero in the directional arguments. Due to their homogeneity in the directional arguments, we have their lower indices. The vectors y_i and y^isatisfy the following relations [27]

a) $y_i = g_{ij}\, y^j$, b) $y_i\, y^i = F^2$, c) $\delta_j^k y^j = y^k$ and d) $y^i_{|k} = 0$. (1.1)

The two sets of quantities g_{ij} and its associate tensor g^{ij} are related by [27].

a) $g_{ij}\, g^{jk} = \delta_i^k = \begin{cases} 1 , & if \quad i = k , \\ 0 , & if \quad i \neq k . \end{cases}$ (1.2)

b) $g^{ij}_{|k} = 0$ and c) $g_{ij|k} = 0$.

The tensor C_{ijk} defined by

$C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\, F^2$, is known as (h) hv-torsion tensor [27]. (1.3)

The (v) hv-torsion tensor C^h_{ik} and its associate (h) hv-torsion tensor C_{ijk} are related by [27].

a) $C^i_{jk}\, y^j = C^i_{kj}\, y^j = 0$ and b) $C_{ijk}\, y^j = 0$. (1.4)

Cartan's covariant derivative of the metric function and the vector y^i vanish identically, i.e.

a) $l^i_{|k} = 0$ and b) $F_{|k} = 0$. (1.5)

The tensor W^i_{jkh} is known as projective curvature tensor (Weyl's projective curvature tensor), the tensor W^i_{jk} is known as projective torsion tensor (Weyl's torsion tensor) and the tensor W^i_j is known as projective deviation tensor (Wely's deviation tensor) are defined by

$$W^i_{jkh} = H^i_{jkh} + \frac{2\delta^i_j}{(n+1)} H_{[hk]} + \frac{2\,y^i}{(n+1)}\dot{\partial}_j H_{[kh]} + \frac{\delta^i_k}{(n^2-1)}\big(n\, H_{jh} + H_{hj} + y^r\dot{\partial}_j H_{hr} - \frac{\delta^i_h}{(n^2-1)}\big(n\, H_{jk} + H_{kj} + y^r\dot{\partial}_j H_{kr}\big)\big) \quad , \tag{1.6}$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{(n+1)} H_{[jk]} + 2\,\{\frac{\delta^i_{[j}}{(n^2-1)}\big(n\, H_{k]} - y^r H_{k]\,r}\big)\} \tag{1.7}$$

And

$$W^i_j = H^i_j - H\,\delta^i_j - \frac{1}{(n+1)}\big(\dot{\partial}_r H^r_j - \dot{\partial}_j H\big)\, y^i \quad , \quad \text{respectively.} \tag{1.8}$$

The tensors W^i_{jkh} , W^i_{jk} and W^i_k are satisfying the following identities

a) $W^i_{jkh}\, y^j = W^i_{kh}$ and b) $W^i_{jk}\, y^j = W^i_k$. (1.9)

The projective curvature tensor W^i_{jkh} is skew-symmetric in its indices k and h.

The Berwald curvature tensor H^i_{jkh} has the torsion tensor H^i_{kh} and deviation tensor H^i_j which arise from the covariant differentiation in the sense of Berwald.

Where the tensors H^i_{jkh} and H^i_{kh} form the components of tensors, and are defined by

a) $H^i_{jkh} = \dot{\partial}_h\, G^i_{jk} + G^r_{jk}\, G^i_{rh} + G^i_{rjh}\, G^r_k - \dot{\partial}_k\, G^i_{jh} - G^r_{jh}\, G^i_{rk} - G^i_{rjk}\, G^r_h$ and

b) $H^i_{kh} = \dot{\partial}_h\, G^i_k + G^r_k\, C^i_{rh} - \dot{\partial}_k\, G^i_h - G^r_h\, C^i_{rk}$. (1.10)

Contraction of the indices i and h in (1.11a) and (1.11b), we get the following

a) $H^i_{jki} = H_{jk}$, b) $H^i_{ki} = H_k$, c) $H^i_{jk}\, y^j = H^i_k$ and d) $H^i_{jk}\, y^k = -H^i_j$. (1.11)

Cartan's third curvature tensor R^i_{jkh} , the R-Ricci tensor R_{jk} in sense of Cartan. The h(v)-torsion tensor H^i_{kh} , the deviation tensor H^i_k , the curvature vector H_k and the scalar curvature H in sense of Berwald, is given by [27].

a) $R^i_{jkh} = \Gamma^{*i}_{hjk} + \left(\Gamma^{*i}_{ljk}\right) G^l_h + C^i_{jm}\left(G^m_{kh} - G^m_{kl} G^l_h\right) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh} - k/h *$, (1.12)

b) $R^i_{jkh}\, y^j = H^i_{kh}$, c) $R_{jk}\, y^j = H_k$, d) $R_{jk}\, y^k = R_j$, e) $R^i_i = R$, f) $R^i_{jki} = R_{jk}$,

g) $H^i_i = (n-1)\, H$, h) $g_{ir} R^i_{jkh} = R_{jrkh}$, i) $g_{ir} R^i_j = R_{jr}$ and j) $R_{jk}\, g^{jk} = R$.

The tensor K^i_{rkj} as defined above is called Cartan's fourth curvature tensor. This tensor is positively homogeneous of degree zero in the directional argument and skew- symmetric in its last two lower indices k and j , i.e.

a) $K^i_{rkj} = \partial_j\, \Gamma^{*i}_{kr} + \left(\dot{\partial}_l\, \Gamma^{*i}_{rj}\right) G^l_k + \Gamma^{*i}_{mj}\, \Gamma^{*m}_{kr} - j/k$ and b) $K^i_{jkh} = -\, K^i_{jhk}$. (1.13)

The curvature tensor K^i_{jkh} satisfies the following relation too

a) $K^i_{jkh}\, y^j = H^i_{kh}$ and b) $H^i_{jkh} = K^i_{jkh} + y^m(\, \dot{\partial}_j\, K^i_{mkh}\,)$. (1.14)

Ricci tensor K_{jk} , curvature vector K_j and curvature scalar K of curvature tensor K^i_{jkh} are given by

a) $K^i_{jki} = K_{jk}$, b) $K_{jk}\, y^k = K_j$, c) $K_{jk}\, y^j = H_k$ and d) $K_{jk}\, g^{jk} = K$. (1.15)

2. On Generalized W^h-Four- Recurrent Space

In a non-flat Finsler space F_n , if there exists a non-zero covariant vectors λ_ℓ and μ_ℓ whose components are positively homogeneous functions of degree zero in $\dot{x}^l$, such that the curvature tensor W^i_{jkh} , which satisfied the following generalized recurrent Finsler space

$$W^i_{jkh|\ell} = \lambda_\ell\, W^i_{jkh} + \mu_\ell\, (\, \delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) \ . \tag{2.1}$$

Where $W^i_{jkh} \neq 0$ and the quantities λ_ℓ and μ_ℓ are non-zero covariant vectors field with respect to $\dot{x}^l$ and we called it the generalized W^h-recurrent space.

From (2.1), the tensor W^i_{jkh} satisfied the following generalized birecurrence condition

$$W^i_{jkh|\ell|m} = a_{\ell m} W^i_{jkh} + b_{\ell m}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) \ . \tag{2.2}$$

Where $|\ell|m$ is h-covariant derivative of second order with respect to x^ℓ and x^m successfully, where $W^i_{jkh} \neq 0$, $a_{\ell m} = (a_{\ell|m} + \lambda_\ell \lambda_m)$ and $b_{\ell m} = (\lambda_\ell\, \mu_m + \mu_{\ell|m})$, the quantities $a_{\ell m}$ and $b_{\ell m}$ are non-zero covariant vectors field and called it generalized W^h-birecurrent space.

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* $-\mathrm{j/k}$ means the subtraction from the former term by interchanging the indices k and h .

Taking h-covariant derivative of (2.2), with respect to x^n and using (1.2c), we get

$$W^i_{jkh|\ell|m|n} = a_{\ell m|n} W^i_{jkh} + a_{\ell m} W^i_{jkh|n} + b_{\ell m|n}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) \quad . \tag{2.3}$$

In view of (2.1), the equation (2.3) yields

$$W^i_{jkh|\ell|m|n} = a_{\ell m|n} W^i_{jkh} + a_{\ell m}\{\lambda_n W^i_{jkh} + \mu_n\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)\} + b_{\ell m|n}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right).$$

(2.4)

Which implies

$$W^i_{jkh|\ell|m|n} = c_{\ell mn} W^i_{jkh} + d_{\ell mn}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) \quad . \tag{2.5}$$

Where $|\ell|m|n$ is h-covariant derivative of third order with respect to x^ℓ , x^m and x^n successfully, $c_{\ell mn} = (a_{\ell|mn} + a_{\ell m}\lambda_n)$ and $d_{\ell mn} = (a_{\ell m}\mu_n + b_{\ell|m|n})$ are non-zero covariant tensors fields of third order, called recurrence tensors field.

3. Certain Generalized of Fourth Order in GW^h-$\mathrm{FRF_n}$

In this section our work in this proposal is defined as Certain generalized tensor of fourth order in GW^h-$\mathrm{FRF_n}$, where $|\ell|m|n|s$ is h-covariant derivative of fourth order in the sense Certain with respect to x^ℓ , x^m, x^n and x^s.

Taking h-covariant derivative of (2.5), with respect to x^s and using (1.2c), we get

$$W^i_{jkh|\ell|m|n|s} = c_{\ell mn|s} W^i_{jkh} + c_{\ell mn} W^i_{jkh|s} + d_{\ell mn|s}(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) \quad . \tag{3.1}$$

In view of (2.1), the above equation can be written as

$$W^i_{jkh|\ell|m|n|s} = c_{lmn|s} W^i_{jkh} + c_{\ell mn}\{\lambda_s W^i_{jkh} + \mu_s\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)\} + d_{lmn|s}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) \quad . \tag{3.2}$$

Which implies

$$W^i_{jkh|\ell|m|n|s} = a_{\ell mns} W^i_{jkh} + b_{\ell mns}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) \quad . \tag{3.3}$$

Definition 3.1. A Finsler space for the tensor W^i_{jkh} is known Weyl's projective curvature tensor and satisfies the condition (3.3), and it will be called generalized four recurrent Finsler space. We shall call a generalized W^h-four-recurrent space and is denoted by GW^h- $\mathrm{FRF_n}$.

Result 3.1. Every generalized W^h-recurrent space is generalized W^h-four recurrent space.

Transvecting (3.3) by y^j using (1.1e), (1.9a) and (1.1a), we get

$$W^i_{kh|\ell|m|n|s} = a_{lmns} W^i_{kh} + b_{lmns}\left(\delta^i_h\, y_k - \delta^i_k\, y_h\right) \quad . \tag{3.4}$$

Again, transvecting (3.4) by y^k using (1.1e), (1.9b) and (1.1b), we get

$$W^i_{h|\ell|m|n|s} = a_{lmns} W^i_h + b_{lmns}(F^2 - 1)\, \delta^i_h \quad . \tag{3.5}$$

Thus, we conclude

Theorem 3.1. In GW^h-FRF_n , Wey'ls projective covariant derivative of forth order for the curvature torsion tensor W^i_{kh} and the curvature deviation tensor W^i_h are given by (3.4) and (3.5), respectively.

Transvecting condition (3.3) by g_{ir} using (1.2c), (1.1d) and $g_{ir}W^i_{jkh} = W_{irkh}$, we get

$$W_{jrkh|\ell|m|n|s} = a_{lmns}W_{jrkh} + b_{lmns}\left(g_{rh}\, g_{jk} - g_{kr}\, g_{jh}\right) . \tag{3.6}$$

Thus, we conclude

Theorem 3.2. In GW^h-FRF_n , the associate curvature tensor W_{irkh} of the projective curvature tensor W^i_{jkh} is given by the condition (3.6).

4. Relation Between Weyl's Curvature Tensor and Other Curvature Tensors

In this section, we shall find the relation between Weyl's curvature tensor W^i_{jkh} and some tensors to be generalized fourth recurrent Finsler space in sense of Cartain.

We know that Weyl's projective curvature tensor W^i_{jkh} and Cartan's four curvature tensor R^i_{jkh} are connected by the formula

$$W^i_{jkh} = R^i_{jkh} + \tfrac{1}{3}\left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h\right) . \tag{4.1}$$

Taking the h-covariant derivative of fourth order for the formula (4.1), x^l, x^m, x^n and x^s, successively, we get

$$W^i_{jkh|\ell|m|n|s} = R^i_{jkh|\ell|m|n|s} + \tfrac{1}{3}\left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h\right)_{|\ell|m|n|s} . \tag{4.2}$$

By using the condition (3.2) and the formula (4.1) in (4.2), we get

$$a_{\ell mns}\left(R^i_{jkh} + \frac{1}{3}\left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h\right)\right) + b_{\ell mns}\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right)$$

$$= R^i_{jkh|\ell|m|n|s} + \tfrac{1}{3}\left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h\right)_{|\ell|m|n|s} . \tag{4.3}$$

Which implies

$$R^i_{jkh|\ell|m|n|s} = a_{\ell mns}R^i_{jkh} + b_{\ell mns}\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) + \tfrac{1}{3}a_{\ell mns}\left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h\right)$$

$$-\tfrac{1}{3}\left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h\right)_{|\ell|m|n|s} . \tag{4.4}$$

This shows that

$$R^i_{jkh|\ell|m|n|s} = a_{\ell mns}R^i_{jkh} + b_{\ell mns}\left(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}\right) . \tag{4.5}$$

If and only if

$$\left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h\right)_{|\ell|m|n|s} = a_{\ell mns}\left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h\right) . \tag{4.6}$$

Thus, we conclude

Theorem 4.1. In $\mathrm{GW^h}$, Cartan's fourth curvature tensor R^i_{jkh} is generalized four recurrent Finsler space if and only if the tensor $\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)$ is four recurrent Finsler space.

Transvecting (4.4) by y^j using (1.1e), (1.12b), (1.12c) and (1.1a), we get

$$H^i_{kh|\ell|m|n|s} = a_{\ell mns} H^i_{kh} + b_{\ell mns}\left(\delta^i_h\, y_k - \delta^i_k\, y_h\right) + \tfrac{1}{3} a_{\ell mns}\left(\delta^i_k H_h - y_k\, R^i_h\right) - \tfrac{1}{3}\left(\delta^i_k H_h - y_k\, R^i_h\right)_{|\ell|m|n|s} . \quad (4.7)$$

This shows that

$$H^i_{kh|\ell|m|n|s} = a_{\ell mns}\, H^i_{kh} + b_{\ell mns}\left(\delta^i_h\, y_k - \delta^i_k\, y_h\right) . \quad (4.8)$$

If and only if

$$\left(\delta^i_k\, H_h - y_k\, R^i_h\right)_{|\ell|m|n|s} = a_{\ell mns}\left(\delta^i_k H_h - y_k\, R^i_h\right) . \quad (4.9)$$

Thus, we conclude

Theorem 4.2. In $\mathrm{GW^h}$-$\mathrm{FRF_n}$, the h-covariant derivative of the four order for the torsion tensor H^i_{kh} (of curvature tensor H^i_{jkh}) is given by the condition (4.8) if and only if the tensor $\left(\delta^i_k H_h - y_k R^i_h\right)$ is four recurrent Finsler space.

Further, transvecting (4.7) by y^k using (1.1e), (1.11c), (1.1b) and (1.1d), we get

$$H^i_{h|\ell|m|n|s} = a_{\ell mns}\, H^i_h + b_{\ell mns}(F^2 - 1)\, \delta^i_h - \tfrac{1}{3} a_{\ell mns}\left(H_h\, y^i - F^2 R^i_h\right) + \tfrac{1}{3}\left(H_h\, y^i - F^2\, R^i_h\right)_{|\ell|m|n|s} . \quad (4.10)$$

This shows that

$$H^i_{h|\ell|m|n|s} = a_{\ell mns}\, H^i_h + b_{\ell mns}(F^2 - 1)\, \delta^i_h . \quad (4.11)$$

If and only if

$$\left(H_h\, y^i - F^2\, R^i_h\right)_{|\ell|m|n|s} = a_{\ell mns}\left(H_h\, y^i - F^2 R^i_h\right) . \quad (4.12)$$

Thus, we conclude

Theorem 4.3. In $\mathrm{GW^h}$-$\mathrm{FRF_n}$, the h-covariant derivative of the four order for the deviation tensor H^i_h is (of Wey'ls projective curvature tensor H^i_{jkh}) is given by the condition (4.11), if and only if the tensor $\left(H_h\, y^i - F^2 R^i_h\right)$ is four recurrent Finsler space.

Contracting the indices i and h in equations (4.7) and (4.11) using (1.11a), (1.11b), (1.2a), (1.1d), (1.1b), (1.12e) and (1.12g), we get

$$H_{k|\ell|m|n|s} = a_{\ell mns} H_k + (n-1) b_{\ell mns}\, y_k + \tfrac{1}{3} a_{\ell mns}(H_k - y_k\, R) - \tfrac{1}{3}(H_k - y_k\, R)_{|\ell|m|n|s} .$$

(4.13)

This shows that

$$H_{k|\ell|m|n|s} = a_{\ell mns} H_k + (n-1) b_{\ell mns}\, y_k \ . \tag{4.14}$$

If and only if

$$(H_k - y_k\, R)_{|\ell|m|n|s} = a_{\ell mns}(H_k - y_k\, R) \tag{4.15}$$

and

$$H_{|\ell|m|n|s} = a_{\ell mns} H + \frac{n}{(n-1)} b_{\ell mns}(F^2 - 1) + \frac{1}{3} a_{\ell mns}\left(H - \frac{1}{(n-1)} F^2 R\right) \tag{4.16}$$
$$-\frac{1}{3}\left(H - \frac{1}{(n-1)} F^2 R\right)_{|\ell|m|n|s} .$$

This shows that

$$H_{|\ell|m|n|s} = a_{\ell mns}\, H + \frac{n}{(n-1)} b_{\ell mns}(F^2 - 1) \ . \tag{4.17}$$

If and only if

$$H_{|\ell|m|n|s} = a_{\ell mns}\, H \ . \tag{4.18}$$

And

$$(F^2 R)_{|\ell|m|n|s} = a_{\ell mns}(F^2 R) \ . \tag{4.19}$$

The equations (4.14) and (4.17) show the curvature vector H_k and the scalar curvature H cannot vanish because the vanishing of any one of them would imply $b_{\ell mns} = 0$; a contradiction.

Thus, we conclude

Theorem 4.4. In GW^h-FRF_n , the curvature vector H_k and the scalar curvature H are given by the conditions (4.14) and (4.17) if and only if the tensors $(H_k - y_k\, R)$, H and $(F^2 R)$ are four recurrent Finsler space, respectively.

Further, transvecting (4.4) by g_{ir} using (1.2c), (1.12h) ,(1.1d) and (1.12i), we get

$$R_{jrkh|\ell|m|n|s} = a_{\ell mns}\, R_{jrkh} + b_{\ell mns}\left(g_{hr} g_{jk} - g_{kr} g_{jh}\right) \tag{4.20}$$
$$+\frac{1}{3} a_{\ell mns}\left(g_{kr} R_{jh} - g_{jk} R_{rh}\right) - \frac{1}{3}\left(g_{kr} R_{jh} - g_{jk} R_{rh}\right)_{|\ell|m|n|s} .$$

This shows that

$$R_{jrkh|\ell|m|n|s} = a_{\ell mns}\, R_{jrkh} + b_{\ell mns}\left(g_{hr} g_{jk} - g_{kr} g_{jh}\right) \ . \tag{4.21}$$

If and only if

$$a_{\ell mns}\left(g_{kr} R_{jh} - g_{jk} R_{rh}\right) = \left(g_{kr} R_{jh} - g_{jk} R_{rh}\right)_{|\ell|m|n|s} \ . \tag{4.22}$$

Thus, we conclude

Theorem 4.5. In GW^h-FRF_n, the associate curvature tensor R_{jpkh} is given by the condition (4.21) if and only if the tensor $\left(g_{kr} R_{jh} - g_{jk} R_{rh}\right)$ is four recurrent Finsler space.

Contracting the indices i and h in (4.4) using (1.12f), (1.2a), (1.1d) and (1.12d), we get

$$R_{jk|\ell|m|n|s} = a_{\ell mns} R_{jk} + (n-1)\, b_{\ell mns}\, g_{jk} + \frac{1}{3} a_{lmns}\left(R_{jk} - g_{jk} R\right) \tag{4.23}$$

$$-\frac{1}{3}\left(R_{jk} - g_{jk}R\right)_{|\ell|m|n|s} \ .$$

This shows that

$$R_{jk|\ell|m|n|s} = a_{\ell mns}R_{jk} + (n-1)\, b_{\ell mns}\, g_{jk} \ . \tag{4.24}$$

If and only if

$$\left(R_{jk} - g_{jk}R\right)_{|\ell|m|n|s} = a_{lmns}\left(R_{jk} - g_{jk}R\right) \ . \tag{4.25}$$

Transvecting (4.23) by g^{jk} using (1.2b), (1.12j) and (1.2a), we get

$$R_{|\ell|m|n|s} = a_{\ell mns}R + n(n-1)\, b_{\ell mns} + \frac{1}{3}a_{lmns}\,(1-n)R \ - \frac{1}{3}\left((1-n)R\right)_{|\ell|m|n|s} . \tag{4.26}$$

This shows that

$$R_{|\ell|m|n|s} = a_{\ell mns}R + a_{lmns}\, n(n-1)\, b_{\ell mns} \ . \tag{4.27}$$

If and only if

$$R_{|\ell|m|n|s} = a_{lmns}\, R \ . \tag{4.28}$$

The equations (4.23) and (4.27) show that the Ricci tensor R_{jk} and the scalar curvature R cannot vanish because the vanishing of any one of them would imply $a_{\ell \mathrm{mns}} = 0$; a contradiction.

Thus, we conclude

Theorem 4.6. In $\mathrm{GW^h\text{-}FRF_n}$, the Ricci tensor R_{jk} and scalar curvature R are non-vanishing if and only if the tensors $\left(R_{jk} - g_{jk}R\right)$ and R are four recurrent Finsler space, respectively.

It is known that Cartan's third curvature tensor R^i_{jkh} and Cartan's fourth curvature tensor K^i_{jkh} are connected by the formula ([16]).

$$R^i_{jkh} = K^i_{jkh} + C^i_{jr}H^r_{hk} \ . \tag{4.29}$$

Using the condition (4.28) in (4.1), we get

$$W^i_{jkh} = K^i_{jkh} + C^i_{jr}H^r_{hk} \ + \frac{1}{3}\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right) . \tag{4.30}$$

Taking the h-covariant derivative of fourth order for the formula (4.29), x^l, x^m, x^n and x^s, successively, we get

$$W^i_{jkh|\ell|m|n|s} = K^i_{jkh|\ell|m|n|s} + \left(C^i_{jr}H^r_{hk}\right)_{|\ell|m|n|s} + \frac{1}{3}\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right)_{|\ell|m|n|s} . \tag{4.31}$$

Form equation (3.3) and using the condition (4.30), we get

$$a_{\ell mns}\left(K^i_{jkh} + C^i_{jr}H^r_{hk} \ + \frac{1}{3}\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right)\right) + b_{\ell mns}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) \tag{4.32}$$

$$= K^i_{jkh|\ell|m|n|s} + \left(C^i_{jr}H^r_{hk}\right)_{|\ell|m|n|s} + \frac{1}{3}\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right)_{|\ell|m|n|s} \ .$$

Which can be written as

$$K^i_{jkh|\ell|m|n|s} = a_{\ell mns}K^i_{jkh} + b_{\ell mns}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + a_{\ell mns}(C^i_{jr}H^r_{hk}) \tag{4.33}$$
$$+\frac{1}{3}a_{\ell mns}(\delta^i_k R_{jh} - g_{jk}R^i_h) - (C^i_{jr}H^r_{hk})_{|\ell|m|n|s} - \frac{1}{3}(\delta^i_k R_{jh} - g_{jk}R^i_h)_{|\ell|m|n|s}.$$

This shows that

$$K^i_{jkh|\ell|m|n|s} = a_{\ell mns}K^i_{jkh} + b_{\ell mns}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) \ . \tag{4.34}$$

If and only if

$$(C^i_{jr}H^r_{hk})_{|\ell|m|n|s} = a_{\ell mns}(C^i_{jr}H^r_{hk}) \tag{4.35}$$

and

$$(\delta^i_k R_{jh} - g_{jk}R^i_h)_{|\ell|m|n|s} = a_{\ell mns}(\delta^i_k R_{jh} - g_{jk}R^i_h) \ .$$

Thus, we conclude

Theorem 4.7. In GW^h , Cartan's fourth curvature tensor K^i_{jkh} is generalized four recurrent Finsler space if and only if the tensors $(C^i_{jr}H^r_{hk})$ and $(\delta^i_k R_{jh} - g_{jk}R^i_h)$ are four recurrent Finsler space, respectively.

Contracting the indices i and h in (4.33) using (1.15a), (1.2a), (1.1d) and (1.12d), we get

$$K_{jk|\ell|m|n|s} = a_{lmns}K_{jk} + b_{lmns}(n-1)\,g_{jk} + a_{lmns}(C^i_{jr}H^r_{ik}) + \frac{1}{3}a_{lmns}(R_{jk} - g_{jk}R)$$
$$-(C^i_{jr}H^r_{ik})_{|\ell|m|n|s} + \frac{1}{3}(R_{jk} - g_{jk}R)_{|\ell|m|n|s} \ . \tag{4.36}$$

This shows that

$$K_{jk|\ell|m|n|s} = a_{lmns}K_{jk} + b_{lmns}(n-1)\,g_{jk} \ . \tag{4.37}$$

If and only if

$$(C^i_{jr}H^r_{ik})_{|\ell|m|n|s} = a_{lmns}(C^i_{jr}H^r_{ik}) \tag{4.38}$$

and

$$(R_{jk} - g_{jk}R)_{|\ell|m|n|s} = a_{lmns}(R_{jk} - g_{jk}R) \ .$$

Thus, we conclude

Theorem 4.8. In GW^h-FRF_n , the Ricci curvature tensor K_{jk} for (Cartan's fourth curvature tensor K^i_{jkh}) is generalized four recurrent Finsler space if and only if the tensors $(C^i_{jr}H^r_{ik})$ and $(R_{jk} - g_{jk}R)$ are four recurrent Finsler space, respectively.

Transvecting (4.35) by y^k using (1.1e), (1.15b), (1.1a), (1.12d) and (1.11d), we get

$$K_{j|\ell|m|n|s} = a_{lmns}K_j + b_{lmns}(n-1)\,y_j - a_{lmns}(C^i_{jr}H^r_i) + \frac{1}{3}a_{lmns}(R_j - y_j R) \tag{4.39}$$
$$+(C^i_{jr}H^r_i)_{|\ell|m|n|s} - \frac{1}{3}(R_j - y_j R)_{|\ell|m|n|s} \ .$$

This shows that

$$K_{j|\ell|m|n|s} = a_{lmns}K_j + b_{lmns}(n-1)\,y_j \ . \tag{4.40}$$

If and only if

$$\left(C^i_{jr}\,H^r_i\right)_{|\ell|m|n|s} = a_{lmns}\left(C^i_{jr}\,H^r_i\right) \tag{4.41}$$

and

$$\left(R_j - y_j\,R\right)_{|\ell|m|n|s} = a_{lmns}\left(R_j - y_j\,R\right).$$

Thus, we conclude

Theorem 4.9. In GW^h-FR F_n , the curvature tensor K_j for (Cartan's fourth curvature tensor K^i_{jkh}) is generalized four recurrent Finsler space if and only if the tensors $\left(C^i_{jr}\,H^r_i\right)$ and $\left(R_j - y_j\,R\right)$ are four recurrent Finsler space, respectively.

5. Conclusions and future work

A Finsler space is called generalized W^h-four recurrent space if it satisfies the condition (3.2). In GW^h-FR F_n , the W^h-covariant derivative of the fourth order for Wely's projective torsion tensor W^i_{kh} and Wely's projective deviation tensor W^i_h are given by (3.4) and (3.5). In GW^h-FR F_n , some necessary and sufficient condition of Cartan's third curvature tensor R^i_{jkh} is generalized four recurrent if and only if the equation (4.6) is good hold, and the h(v)-torsion tensor H^i_{kh} is generalized four recurrent if and only if the equation (4.8) is good hold. In GW^h-FR F_n , we get the same relationship between Wely's projective torsion tensor and the tensors R^i_{jkh} and K^i_{jkh}. In GW^h-FR F_n, Cartan's fourth curvature tensor K^i_{jkh} is generalized four recurrent if and only if the tensors $\left(C^i_{jr}H^r_{hk}\right)$ and $\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right)$ are four recurrent Finsler space. W^h-covariant derivative of the fourth order for the K-Ricci tensor K_{jk} is generalized four recurrent if and only if the tensors $\left(C^i_{jr}H^r_{ik}\right)$ and $\left(R_{jk} - g_{jk}R\right)$ are four recurrent Finsler space.

Authors recommend the need for continuing research and development in generalized W^h-five recurrent Finsler spaces and interlard it with the properties of special spaces for Finsler space.

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العلاقة بين الموتر الأسقاطي لوايل وبعض المنحنيات الاخرى

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الملخص

في هذه الورقة قدم الباحثان التحقق من العلاقة بين الموتر الأسقاطي لوايل وبعض المنحنيات الاخرى حيث ان الموتر W^i_{jkh} يحقق تعميم خاصية المشتقة الرابعة بالنسبة لاشتقاق كارتان وهذا النوع من الفضاء أطلقنا عليه تعميم فضاء فنسلر W^h – رباعي الاشتقاق ورمزنا إليه بالرمز التالي $GW^h\text{-}FRF_n$.

لائحة بالرموز

W^i_{jkh} : *منحنى الموتر الأسقاطي لوايل*

R^i_{jkh} : *منحنى الموتر الثالث لكارتان*

P^i_{jkh} : *منحنى الموتر الثاني لكارتان*

K^i_{jkh} : *منحنى الموتر الرابع لكارتان*

H^i_{jkh} : *منحنى الموتر لباروالد*

W_{jk} : *منحنى ريتشي*

W_k : *منحنى المتجه*

W_{jrkh} : *منحنى الموتر المساعد ل* W^i_{jkh}

معلومات البحث

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الكلمات المفتاحية

فضاء فنسلر F_n، تعميم فضاء فنسلر– W^h رباعي الاشتقاق، مشتقة كارتان من الرتبة الرابعة، الموتر الأسقاطي لوايل W^i_{jkh} ، الموتر التقوسي الثالث لكارتان R^i_{jkh} .