Extension of Exton's Quadruple Hypergeometric Function K14

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Abstract
The main object of this paper is to introduce an extension of Exton's quadruple hypergeometric function K_{14} by using the extended Euler's beta function obtained earlier by Özergin, Özarslanand and Altin (2011). For this extended function, we investigate various properties such as integral representations, recurrence relations, generating functions, transformation formulas and summation formulas. Some special cases of the main results of this paper are also considered.

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1. Introduction

Exton (1976) defined the quadruple hypergeometric function K_{14} by the following series:

$$K_{14}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,t)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p}(c_3)_q(b)_{m+q}(c_1)_n(c_2)_p}{(d)_{m+n+p+q}}\frac{x^m y^n z^p t^q}{m!\,n!\,p!\,q!}, \quad (1)$$

where $(a)_n=\dfrac{\Gamma(a+n)}{\Gamma(a)}=a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol.

The following integral representations of the function K_{14} is also given by Exton (1976):

$$K_{14}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,t)$$
$$=\frac{\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(d-a-b)}\int_0^{\infty}\int_0^{\infty}u^{a-1}v^{b-1}(1-u)^{d-a-b}(1-v)^{d-a-b-1}$$
$$\times(1-ux)^{c_3-b}(1-uy)^{-c_1}(1-uz)^{-c_2}(1-ux-vt+uvt)^{-c_3}\,du\,dv, \quad (2)$$

where $\mathrm{Re}(a)>0,\ \mathrm{Re}(b)>0,\ \mathrm{Re}(d-a-b)>0,\ |x|+|t|<1,\ |y|<1\ and\ |z|<1.$

Özergin et al. (2011) introduced the following extended beta function:

$$B_p^{(\alpha,\beta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt, \quad (3)$$

$$(\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(p)>0,\ \mathrm{Re}(x)>0,\ \mathrm{Re}(y)>0).$$

The special case of (3) when $p=0$, yields the classical beta function $B(x,y)$ (see Srivastava and Manocha (1984))

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\ \mathrm{Re}(x)>0,\ \mathrm{Re}(y)>0. \quad (4)$$

Using the extended beta function given in (3), Özergin et al. (2011) introduced the following extended Gauss hypergeometric function:

$$F_p^{(\alpha,\beta)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \quad (5)$$

$$(\mathrm{Re}(c)>\mathrm{Re}(b)>0,\ |z|<1)$$

and obtained the following integral representation for the extended Gauss hypergeometric function given in (5):

$$F_p^{(\alpha,\beta)}(a,b;c;z)$$
$$=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt, \quad (6)$$

$$(\mathrm{Re}(p)>0;\ p=0\ \mathit{and}\ |\arg(1-z)|<\pi;\ \mathrm{Re}(c)>\mathrm{Re}(b)>0).$$

The special case of (5) when $p=0$ yields the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ (see Srivastava and Manocha (1984))

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\qquad c\neq 0,-1,-2,\cdots. \quad (7)$$

Recently various extensions of some hypergeometric functions of two and three variables are considered (see Çetinkaya et al., 2016; Liu, 2014; Özarslan and Özergin, 2010; Shadab and Choi, 2017). In terms of the extended beta function given in (3), Liu (2014) defined the extended Appell's function $F_{1,p}^{(\alpha,\beta)}$ and the extended Lauricella's function $F_{D,p}^{(3,\alpha,\beta)}$ as follows:

$$F_{1,p}^{(\alpha,\beta)}(a,b,c;d;x,y)=\sum_{m,n=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n,d-a)(b)_m(c)_n}{B(a,d-a)}\frac{x^m}{m!}\frac{y^n}{n!} \quad (8)$$

$$(Max\{|x|,|y|\}<1;\ \mathrm{Re}(p)\geq 0)$$

and

$$F_{D,p}^{(3,\alpha,\beta)}(a,b,c,d;e;x,y,z)$$
$$=\sum_{m,n,r=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n+r,e-a)(b)_m(c)_n(d)_r x^m y^n z^r}{B(a,e-a)\ m!n!r!} \quad (9)$$

$$(Max\{|x|,|y|\}<1;\ \mathrm{Re}(p)\geq 0).$$

The following integral representation of the extended Appell's function $F_{1,p}^{(\alpha,\beta)}$ and the extended Lauricella's function $F_{D,p}^{(3,\alpha,\beta)}$ are also given by Liu (2014):

$$F_{1,p}^{(\alpha,\beta)}(a,b,c;d;x,y) = \frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt \quad (10)$$

and

$$F_{D,p}^{(3,\alpha,\beta)}(a,b,c,d;e;x,y,z) = \frac{1}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c}(1-zt)^{-d}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt. \quad (11)$$

The special case of (8) and (9) when $p=0$ yields, respectively, Appell function F_1 and Lauricella function $F_D^{(3)}$ (see Srivastava and Manocha (1984))

$$F_1(a,b,c;d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \quad (12)$$

and

$$F_D^{(3)}(a,b,c,d;e;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}(b)_m(c)_n(d)_p}{(e)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}. \quad (13)$$

2. Extended Exton's Hypergeometric Function $K_{14,p}^{(\alpha,\beta)}$

Here, we use the extended beta function given in (3) to define the extended Exton's hypergeometric function $K_{14,p}^{(\alpha,\beta)}$ as follows:

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,u) = \sum_{m,n,r,s=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(a+m+n+r,d-a+s)(b)_{m+s}(c_1)_n(c_2)_r(c_3)_s\, x^m y^n z^r u^s}{B(a,d-a)(d-a)_s\, m!n!r!s!}. \quad (14)$$

The extended Exton's hypergeometric function $K_{14,p}^{(\alpha,\beta)}$ given in (14) can be written as follows:

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,u) = \sum_{s=0}^{\infty} \frac{(c_3)_s(b)_s}{(d)_s} F_{D,p}^{(3;\alpha,\beta)}(a,b+s,c_1,c_2;d+s;x,y,z)\frac{u^s}{s!}. \quad (15)$$

Remark 2.1. The special case $c_3 = d-a$ of (14) yields the following extended Exton's hypergeometric function $K_{14,p}^{(\alpha,\beta)}$:

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,d-a;b,c_1,c_2,b;d,d,d,d;x,y,z,u) = \sum_{m,n,r,s=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(a+m+n+r,d-a+s)(b)_{m+s}(c_1)_n(c_2)_r\, x^m y^n z^r u^s}{B(a,d-a)\, m!n!r!s!}. \quad (16)$$

Remark 2.2. The case $p=0$ of (14) yields the original function K_{14} given in (1)

$$K_{14,0}^{(\alpha,\beta)}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,u) = K_{14}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,u). \qquad (17)$$

3. Integral Representations

Theorem 3.1. The following integral representations for the extended Exton's hypergeometric function $K_{14,p}^{(\alpha,\beta)}$ hold true:

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,u) = \frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c_1}(1-zt)^{-c_2} \times {}_2F_1\left(c_3,b;d-a;\frac{u(1-t)}{1-xt}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt \qquad (18)$$

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,u) = \frac{1}{B(a,d-a)B(b,d-a-b)} \times \int_0^1\int_0^1 t^{a-1}s^{b-1}(1-t)^{d-a-1}(1-s)^{d-a-b-1}(1-xt)^{-c_3-b}(1-yt)^{-c_1}(1-zt)^{-c_2} \times (1-xt-us+ust)^{-c_3}\ {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dtds \qquad (19)$$

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,u) = \frac{1}{B(a,d-a)B(b,d-a-b)} \int_0^1\int_0^1 t^{a-1}s^{b-1}(1-t)^{d-a-1}(1-s)^{d-a-b-1}(1-yt)^{-c_1}(1-zt)^{-c_2} \times(1-us)^{-c_3}\left(1-\left(\frac{x-us}{1-us}\right)t\right)^{-c_3}\ {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dtds \qquad (20)$$

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,u) = \frac{2}{B(a,d-a)}\int_0^{\frac{\pi}{2}} \sin^{2a-1}\theta\cos^{2d-2a-1}\theta(1-x\sin^2\theta)^{-b}(1-y\sin^2\theta)^{-c_1}(1-z\sin^2\theta)^{-c_2} \times {}_2F_1\left(b,c_3;d-a;\frac{u\cos^2\theta}{1-x\sin^2\theta}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{\sin^2\theta\cos^2\theta}\right)d\theta \qquad (21)$$

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,u) = \frac{1}{B(a,d-a)}\int_0^{\infty} \xi^{a-1}(1+\xi)^{b+c_1+c_2-d}(1+(1-x)\xi)^{-b}(1+(1-y)\xi)^{-c_1}(1+(1-z)\xi)^{-c_2} \times {}_2F_1\left(b,c_3;d-a;\frac{u}{1+(1-x)\xi}\right){}_1F_1\left(\alpha;\beta;-\frac{p(1+\xi)^2}{\xi}\right)d\xi. \qquad (22)$$

***Proof.* of (18).** Using (3) on the right-hand side of (14) and interchanging the order of summation and integration, we have:

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,u)$$

$$=\frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}\sum_{s=0}^{\infty}\frac{(b)_s(c_3)_s(u(1-t))^s}{(d-a)_s\, s!}\left(\sum_{m=0}^{\infty}\frac{(b+s)_m (xt)^m}{m!}\right)$$

$$\times\left(\sum_{n=0}^{\infty}\frac{(c_1)_n (yt)^n}{n!}\right)\left(\sum_{r=0}^{\infty}\frac{(c_2)_r (zt)^r}{r!}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt,$$

which on using the following result (Srivastava and Manocha (1984)):

$$\sum_{n=0}^{\infty}(a)_n\frac{x^n}{n!}=(1-x)^{-a}, \tag{23}$$

yields the desired result (18)**.** The integral representation (19) can be obtained easily from (18) by using the following integral representations of ${}_2F_1(a,b;c;x)$ (Srivastava and Manocha, 1984):

$${}_2F_1(a,b;c;x)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt\ . \tag{24}$$

Also, the integral representation (20) can be obtained directly from (18) if we use the following relation:

$$(1-xt-z(1-t))^{-a}=(1-z)^{-a}\left(1-\frac{(x-z)t}{1-z}\right)^{-a}. \tag{25}$$

Finally, the integral representations (21) and (22) can be easily obtained by taking the transformations $t=\sin^2\theta$ and $t=\dfrac{\xi}{1+\xi}$ in (18) respectively. This completes the proof of Theorem 3.1.

Remark 3.1. The special case $c_3=d-a$ of (18), (21) and (22) yields the following results:

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,d-a;b,c_1,c_2,b;d,d,d,d;x,y,z,u)$$

$$=\frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt-u(1-t))^{-b}(1-yt)^{-c_1}(1-zt)^{-c_2}$$

$$\times{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt, \tag{26}$$

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,d-a;b,c_1,c_2,b;d,d,d,d;x,y,z,u)$$

$$=\frac{2}{B(a,d-a)}\int_0^{\frac{\pi}{2}}\sin^{2a-1}\theta\cos^{2d-2a-1}\theta(1-x\sin^2\theta-u\cos^2\theta)^{-b}(1-y\sin^2\theta)^{-c_1}$$

$$\times(1-z\sin^2\theta)^{-c_2}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{\sin^2\theta\cos^2\theta}\right)d\theta \tag{27}$$

and

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,d-a;b,c_1,c_2,b;d,d,d,d;x,y,z,u)$$

$$=\frac{1}{B(a,d-a)}\int_0^{\infty}\xi^{a-1}(1+\xi)^{b+c_1+c_2-d}(1+(1-x)\xi-u)^{-b}(1+(1-y)\xi)^{-c_1}$$

$$\times(1+(1-z)\xi)^{-c_2}\,{}_1F_1\left(\alpha;\beta;-\frac{p(1+\xi)^2}{\xi}\right)d\xi\,. \tag{28}$$

Remark 3.2. The case $p=0$ of (19) yields the integral representation given in (2).

4. Recurrence Relations

Theorem 4.1. The following recurrence relations for the extended Exton's hypergeometric function $K_{14,p}^{(\alpha,\beta)}$ holds true:

(i) $$(\beta-\alpha)K_{14,p}^{(\alpha-1,\beta)} - \alpha K_{14,p}^{(\alpha+1,\beta)} + (2\alpha-\beta)K_{14,p}^{(\alpha,\beta)} - \frac{pB(a-1,e-a-1)}{B(a,e-a)} \times K_{14,p}^{(\alpha,\beta)}\left(a-1,a-1,a-1,e-a-1;b,c,d,b;e-2,e-2,e-2,e-2;x,y,z,u\right)=0 \quad (29)$$

(ii) $$\beta K_{14,p}^{(\alpha,\beta)} - \beta K_{14,p}^{(\alpha-1,\beta)} + \frac{pB(a-1,e-a-1)}{B(a,e-a)} \times K_{14,p}^{(\alpha,\beta+1)}\left(a-1,a-1,a-1,e-a-1;b,c,d,b;e-2,e-2,e-2,e-2;x,y,z,u\right)=0 \quad (30)$$

(iii) $$(\beta-\alpha-1)K_{14,p}^{(\alpha,\beta)} + \alpha K_{14,p}^{(\alpha+1,\beta)} - (\beta-1)K_{14,p}^{(\alpha,\beta-1)} = 0 \quad (31)$$

(iv) $$(\alpha-1)K_{14,p}^{(\alpha,\beta)} + (\beta-\alpha)K_{14,p}^{(\alpha-1,\beta)} - (\beta-1)K_{14,p}^{(\alpha,\beta-1)} - \frac{pB(a-1,e-a-1)}{B(a,e-a)} \times K_{14,p}^{(\alpha,\beta)}\left(a-1,a-1,a-1,e-a-1;b,c,d,b;e-2,e-2,e-2,e-2;x,y,z,u\right)=0\,, \quad (32)$$

where $K_{14,p}^{(\alpha,\beta)} = K_{14,p}^{(\alpha,\beta)}\left(a,a,a,e-a;b,c,d,b;e,e,e,e;x,y,z,u\right)$.

Proofs. To prove our results in Theorem 4.1, we require the following recurrence relations of the confluent function ${}_1F_1$ (Luke, Y. L., 1969):

$$(\beta-\alpha)\ {}_1F_1\left(\alpha-1;\beta;z\right) - \alpha\ {}_1F_1\left(\alpha+1;\beta;z\right) + (2\alpha-\beta+z)\ {}_1F_1\left(\alpha;\beta;z\right) = 0 \quad (33)$$

$$\beta\ {}_1F_1\left(\alpha;\beta;z\right) - \beta\ {}_1F_1\left(\alpha-1;\beta;z\right) - z\ {}_1F_1\left(\alpha;\beta+1;z\right) = 0 \quad (34)$$

$$(\beta-\alpha-1)\ {}_1F_1\left(\alpha;\beta;z\right) + \alpha\ {}_1F_1\left(\alpha+1;\beta;z\right) - (\beta-1)\ {}_1F_1\left(\alpha;\beta-1;z\right) = 0 \quad (35)$$

$$(\alpha+z-1)\ {}_1F_1\left(\alpha;\beta;z\right) + (\beta-\alpha)\ {}_1F_1\left(\alpha-1;\beta;z\right) - (\beta-1)\ {}_1F_1\left(\alpha;\beta-1;z\right) = 0\,. \quad (36)$$

Proof. **of (29).** Replacing z by $-\frac{p}{t(1-t)}$ in (33), multiplying both sides by $t^{a-1}(1-t)^{e-a-1}(1-xt-u(1-t))^{-b}(1-yt)^{-c}(1-zt)^{-d}/B(a,e-a)$ and integrating the resultant equation with respect to t between the limits 0 to 1, we get

$$\frac{\beta-\alpha}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt-u(1-t))^{-b}(1-yt)^{-c}(1-zt)^{-d}\ {}_1F_1\left(\alpha-1;\beta;-\tfrac{p}{t(1-t)}\right)dt$$
$$-\frac{\alpha}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt-u(1-t))^{-b}(1-yt)^{-c}(1-zt)^{-d}\ {}_1F_1\left(\alpha+1;\beta;-\tfrac{p}{t(1-t)}\right)dt$$
$$+\frac{2\alpha-\beta}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt-u(1-t))^{-b}(1-yt)^{-c}(1-zt)^{-d}\ {}_1F_1\left(\alpha;\beta;-\tfrac{p}{t(1-t)}\right)dt$$
$$-\frac{p}{B(a,e-a)}\int_0^1 t^{a-2}(1-t)^{e-a-2}(1-xt-u(1-t))^{-b}(1-yt)^{-c}(1-zt)^{-d}\ {}_1F_1\left(\alpha;\beta;-\tfrac{p}{t(1-t)}\right)dt=0,$$

which on using the integral representation (26), yields the desired result (29).
The results (30), (31) and (32) can be proved by a similar method as in the proof of (29) and we use here the recurrence relations (34), (35) and (36).

5. Generating Functions

Theorem 5.1. The following generating functions for the extended Exton's hypergeometric function $K_{14,p}^{(\alpha,\beta)}$ hold true:

$$\sum_{k=0}^{\infty}\frac{(c_3)_k\ t^k}{k!}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3+k;b,c_1,c_2,b;d,d,d,d;x,y,z,u\right)$$
$$=(1-t)^{-c_3}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,z,\frac{u}{1-t}\right) \tag{37}$$

$$\sum_{k=0}^{\infty}\frac{(b)_k\ t^k}{k!}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3;b+k,c_1,c_2,b+k;d,d,d,d;x,y,z,u\right)$$
$$=(1-t)^{-b}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;\frac{x}{1-t},y,z,\frac{u}{1-t}\right) \tag{38}$$

$$\sum_{k=0}^{\infty}\frac{(c_1)_k\ t^k}{k!}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3;b,c_1+k,c_2,b;d,d,d,d;x,y,z,u\right)$$
$$=(1-t)^{-c_1}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,\frac{y}{1-t},z,u\right) \tag{39}$$

$$\sum_{k=0}^{\infty}\frac{(c_2)_k\ t^k}{k!}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3;b,c_1,c_2+k,b;d,d,d,d;x,y,z,u\right)$$
$$=(1-t)^{-c_2}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;x,y,\frac{z}{1-t},u\right). \tag{40}$$

***Proof.* of (37).** Using (14) in the L.H.S. of equation (37), we get

$$\sum_{k=0}^{\infty}\frac{(c_3)_k\ t^k}{k!}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3+k;b,c_1,c_2,b;d,d,d,d;x,y,z,u\right)$$
$$=\sum_{m,n,r,s,k=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n+r,d-a+s)(c_3)_{s+k}(b)_{m+s}(c_1)_n(c_2)_r\,x^m y^n z^r u^s t^k}{B(a,d-a)(d-a)_s\,m!n!r!s!k!}$$
$$=\sum_{m,n,r,s=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n+r,d-a+s)(c_3)_s(b)_{m+s}(c_1)_n(c_2)_r\,x^m y^n z^r u^s}{B(a,d-a)(d-a)_s\,m!n!r!s!}\sum_{k=0}^{\infty}\frac{(c_3+s)_k t^k}{k!},$$

which on using (23), we obtain the desired result (37)**.**

The generating functions (38), (39) and (40) can be proved by a similar method as in the proof of (37).

Remark 5.1. Setting $p=0$ in (37), (38), (39) and (40), we get a known results of Chandel and Tiwari (1991).

Theorem 5.2. The following generating functions for the extended Exton's hypergeometric function $K_{14,p}^{(\alpha,\beta)}$ hold true:

$$\sum_{k=0}^{\infty}\frac{(\lambda)_k\ t^k}{k!}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,-k;b,c_1,c_2,b;d,d,d,d;x,y,z,u\right)$$
$$=(1-t)^{-\lambda}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,\lambda;b,c_1,c_2,b;d,d,d,d;x,y,z,\frac{-ut}{1-t}\right) \tag{41}$$

$$\sum_{k=0}^{\infty}\frac{(\lambda)_k\, t^k}{k!}K_{14,p}^{(\alpha,\beta)}(a,a,a,c_3;-k,c_1,c_2,-k;d,d,d,d;x,y,z,u)$$
$$=(1-t)^{-\lambda}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3;\lambda,c_1,c_2,\lambda;d,d,d,d;\frac{-xt}{1-t},y,z,\frac{-ut}{1-t}\right) \qquad (42)$$

$$\sum_{k=0}^{\infty}\frac{(\lambda)_k\, t^k}{k!}K_{14,p}^{(\alpha,\beta)}(a,a,a,c_3;b,-k,c_2,b;d,d,d,d;x,y,z,u)$$
$$=(1-t)^{-\lambda}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3;b,\lambda,c_2,b;d,d,d,d;x,\frac{-yt}{1-t},z,u\right) \qquad (43)$$

$$\sum_{k=0}^{\infty}\frac{(\lambda)_k\, t^k}{k!}K_{14,p}^{(\alpha,\beta)}(a,a,a,c_3;b,c_1,-k,b;d,d,d,d;x,y,z,u)$$
$$=(1-t)^{-\lambda}K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3;b,c_1,\lambda,b;d,d,d,d;x,y,\frac{-zt}{1-t},u\right). \qquad (44)$$

***Proof*. of (41).** In the L.H.S. of equation (41) expressing $K_{14,p}^{(\alpha,\beta)}$ as in (14) and using the results (Srivastava and Manocha, 1984)

$$(-n)_k=\frac{(-1)^k\, n!}{(n-k)!},\qquad 0\le k\le n,\quad \sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+k)\ , \qquad (45)$$

we obtain:

$$\sum_{k=0}^{\infty}\frac{(\lambda)_k\, t^k}{k!}K_{14,p}^{(\alpha,\beta)}(a,a,a,-k;b,c_1,c_2,b;d,d,d,d;x,y,z,u)$$
$$=\sum_{m,n,r,s,k=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n+r,d-a+s)\,(\lambda)_{k+s}(b)_{m+s}(c_1)_n(c_2)_r x^m y^n z^r(-u)^s t^{k+s}}{B(a,d-a)(d-a)_s\qquad m!\,n!\,r!\,s!\,k!}$$
$$=\sum_{m,n,r,s,k=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n+r,d-a+s)\,(\lambda)_{s}(b)_{m+s}(c_1)_n(c_2)_r x^m y^n z^r(-ut)^s}{B(a,d-a)(d-a)_s\qquad m!\,n!\,r!\,s!\,k!}$$
$$\times\sum_{k=0}^{\infty}\frac{(\lambda+s)_k t^k}{k!},$$

which on using (23), we obtain the desired result (41) **.**

The generating functions (42), (43) and (44) can be proved by a similar method as in the proof of (41).

6. Transformation and Summation Formulas

Theorem 6.1. The following transformation formula for the extended Exton's hypergeometric function $K_{14,p}^{(\alpha,\beta)}$ holds true:

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,d-a;b,c_1,c_2,b;d,d,d,d;x,y,z,u)$$
$$=(1-u)^{-b}F_{D,p}^{(3,\alpha,\beta)}\left(a,b,c_1,c_2;d;\frac{x-u}{1-u},y,z\right). \qquad (46)$$

***Proof*.** Using (25) in (26), we have

$$K_{14,p}^{(\alpha,\beta)}(a,a,a,d-a;b,c_1,c_2,b;d,d,d,d;x,y,z,u)$$
$$=\frac{(1-u)^{-b}}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}\left(1-\left(\frac{x-u}{1-u}\right)t\right)^{-b}(1-yt)^{-c_1}(1-zt)^{-c_2}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt\,,$$

which by using (11), yields the desired result (46).

Remark 6.1. The special case $u=x$ of (46) yields the following result:

$$K_{14,p}^{(\alpha,\beta)}\left(a,a,a,d-a;b,c_1,c_2,b;d,d,d,d;x,y,z,x\right) = (1-x)^{-b}\ F_{1,p}^{(\alpha,\beta)}\left(a,c_1,c_2;d;y,z\right). \qquad (47)$$

Remark 6.2. For $p=0$, equation (46) reduces to a known result Exton, H. (1976)

$$K_{14}\left(a,a,a,d-a;b,c_1,c_2,b;d,d,d,d;x,y,z,u\right) = (1-u)^{-b}\ F_D^{(3)}\left(a,b,c_1,c_2;d;\frac{x-u}{1-u},y,z\right). \qquad (48)$$

Theorem 6.2. The following summation formulas for the extended Exton's hypergeometric function $K_{14,p}^{(\alpha,\beta)}$ holds true:

$$K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;1,1,1,1\right) = \frac{\Gamma(d)\Gamma(d-a-b-c_3)}{\Gamma(a)\Gamma(d-a-b)\Gamma(d-a-c_3)} B_p^{(\alpha,\beta)}(a,d-a-b-c_1-c_2) \qquad (49)$$

$$K_{14,p}^{(\alpha,\beta)}\left(a,a,a,1-b;b,c_1,c_2,b;d,d,d,d;1,1,1,\tfrac{1}{2}\right) = \frac{\Gamma(d)\Gamma(\frac{1}{2}d-\frac{1}{2}a)\Gamma(\frac{1}{2}d-\frac{1}{2}a+\frac{1}{2})B_p^{(\alpha,\beta)}(a,d-a-b-c_1-c_2)}{\Gamma(a)\Gamma(d-a)\Gamma(\frac{1}{2}d-\frac{1}{2}a+\frac{1}{2}b)\Gamma(\frac{1}{2}d-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}. \qquad (50)$$

Proof of (**49).** Setting $x=y=z=u=1$ in (18) and using the following formula

Rainville (1960):

$${}_2F_1\left(a,b;c;1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad (51)$$

we get:

$$K_{14,p}^{(\alpha,\beta)}\left(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;1,1,1,1\right) = \frac{\Gamma(d)\Gamma(d-a-b-c_3)}{\Gamma(a)\Gamma(d-a-b)\Gamma(d-a-c_3)}\int_0^1 t^{a-1}\,(1-t)^{d-a-b-c_1-c_2-1}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt, \qquad (52)$$

which by using (3), yields the desired result (49).
The proof of (50) is similar to that of (49) and we use the following formula (Rainville, 1960):

$${}_2F_1\left(a,1-a;c;\tfrac{1}{2}\right) = \frac{\Gamma(\frac{1}{2}c)\Gamma(\frac{1}{2}c+\frac{1}{2})}{\Gamma(\frac{1}{2}c+\frac{1}{2}a)\Gamma(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2})}. \qquad (53)$$

Remark 6.3. Setting $p=0$ in (49) and (50), we get respectively the following known summation formulas of Exton's function K_{14} (Atash and Bellehaj, 2020):

$$K_{14}\left(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;1,1,1,1\right) = \frac{\Gamma(d)\Gamma(d-a-b-c_3)\Gamma(d-a-b-c_1-c_2)}{\Gamma(d-a-b)\Gamma(d-a-c_3)\Gamma(d-b-c_1-c_2)} \qquad (54)$$

and

$$K_{14}\left(a,a,a,1-b;b,c_1,c_2,b;d,d,d,d;1,1,1,\tfrac{1}{2}\right)$$

$$= \frac{\Gamma(d)\Gamma(d-a-b-c_1-c_2)\Gamma(\frac{1}{2}d-\frac{1}{2}a)\Gamma(\frac{1}{2}d-\frac{1}{2}a+\frac{1}{2})}{\Gamma(d-a)\Gamma(d-b-c_1-c_2)\Gamma(\frac{1}{2}d-\frac{1}{2}a+\frac{1}{2}b)\Gamma(\frac{1}{2}d-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}. \quad (55)$$

7. Conclusion

In the present paper, we have extended the known Exton's quadruple hypergeometric function K_{14} by using the extended Euler's beta function obtained earlier by Özergin et al. (2011). For this new extended function $K_{14,p}^{(\alpha,\beta)}$, we have obtained various properties such as integral representations, recurrence relations, generating functions, transformation formulas and summation formulas. Furthermore, some known results for the quadruple hypergeometric function K_{14} are also given as special cases of our main formulas. The method used in this paper can be applied to extend many other hypergeometric functions given in the literature.

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تمديد دالة اكستون الرباعية الفوق هندسية K_{14}

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الملخص

الهدف الأساسي لبحثنا هذا هو إدخال تمديد جديد لدالة اكستون الرباعية الفوق هندسية K_{14} باستخدام تمديد دالة اويلر بيتا المعطى سابقاً بواسطة الباحث Özergin, Özarslanand and Altin (2011) قمنا بمناقشة واستنتاج العديد من خواص هذه الدالة الجديدة مثل التمثيلات التكاملية والعلاقات التكرارية والدوال المولدة والصيغ التحويلية والجمعية، وتم أيضاً عرض العديد من النتائج المعروفة سابقاً وذلك كحالات خاصة لنتائج بحثنا الرئيسية.

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دالة اكستون الممدة، دالة بيتا الممدة، التمثيلات التكاملية، العلاقات التكرارية، الدوال المولدة، الصيغ التحويلية، الصيغ الجمعية