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Persistent Homology in the Analysis of Electroencephalography Signals during Epileptic Seizures: Visualizing Topological Patterns and Dynamics

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Abstract
Electroencephalography, commonly known as EEG, is a valuable and preferred tool for diagnosing various brain disorders, particularly epilepsy, due to its non-invasive nature and ability to provide comprehensive insights into brain function through electrical potentials. However, the results of EEG assessments during epileptic seizures are often perceived as noise rather than a structured pattern. EEG signals during an epileptic seizure can be viewed as an integral equation, and the algebraic topological structure provides tools to analyse the shape and structure of EEG signal data by applying concepts such as homology and persistent homology. In this paper, we demonstrate the application of persistent homology in the analysis of EEG signals during epileptic seizures by constructing an ordered topological and dynamical representation of the EEG signals as an integral equation.

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EEG signals, integral equation, homology, topological data analysis

1. Introduction

Epilepsy is a chronic neurological disorder that affects individuals worldwide. It is defined by the generation of recurrent seizures due to sudden, usually transient, bursts of excessive electrical activity occurring in specific groups of brain cells. Seizures are divided into two main forms: seizures episodic and generalized seizures (Beghi, 2020). A partial seizure arises when clinical or electroencephalographic evidence suggests that the incoming information originates from an area in the brain (Tatum et al., 2018). These seizures initially affect only a portion of the brain, causing symptoms in the corresponding areas of the body or impairment of related cognitive functions unlike a generalized seizure, which occurs when evidence suggests that electrical currents in the brain are more widespread (Patel & Moshé, 2020).

EEG, which is an important diagnostic tool, comes in handy in the diagnosis and management of epilepsy. An EEG measures the electrical activity generated by firing neurons in the brain (McInnis et al., 2023). It works by recording changes in electrical fields recorded by electrodes located on the patient’s scalp, these changes reflect neural activity in advanced computer systems such as the Nicolet-One which are used to convert EEG signals into digital form. So for subsequent statistical analysis, it is common to hypothesize that the mathematical analysis of existing EEG signals helps healthcare professionals by providing a detailed description of the observed brain activity, thus making sense of human cerebral cortex is increased (Najafi et al., 2022).

Ahmed et al. 2000 introduced a fuzzy-based topological model aimed at identifying the sources of epileptic seizures. Called Fuzzy Topographic Topological Mapping (FTTM), this model applies a topological profile to the magnetic fields captured in magnetoencephalographic (MEG) recordings, using fuzzy logic methods to map the locations of epileptic foci—specific areas in the brain (T. Ahmad et al., 2000). This approach was improved in 2008 by including EEG signal data in the analysis (T. Ahmad et al., 2008). Beyond merely identifying foci locations, the FTTM method also tracks important events throughout the progression of seizures, as evidenced in Idris's 2010 study (A. Idris et al., 2010). However, the model has faced criticism for potential information loss during the fuzzification and defuzzification phases. Subsequently, in 2024, Ameen developed a novel mathematical model that addresses these ambiguities in estimating seizure foci by employing integral equations, effectively characterizing EEG signals during seizures within this mathematical framework (Barja, 2024).

2. Related Work and Preliminary Considerations

In this section, we will explore the existing literature relevant to our research, highlighting key studies and findings that inform our work while also presenting fundamental definitions and essential theorems that underpin the concepts discussed. By examining prior research and establishing a clear theoretical framework, we aim to provide a comprehensive context for our study and clarify the significance of our contributions to the field. Additionally, as mentioned in the introduction, Ameen developed a new model to describe EEG signals during an epileptic seizure as an integral equation of the following form:

$$\phi(t) = \int_{t}^{\infty} K(\tau)\, v(t-\tau)\mu(\tau)d\tau \qquad (2.1)$$

The equation (2.1), with its kernel function $K(\tau)$, pre-seizure signal $v(t-\tau)$, and seizure function $\mu(\tau)$, served as a key tool for unlocking the connection between the observed EEG signal $\phi(t)$ during a seizure and the underlying brain activity $\mu(\tau)$. By solving (2.1), we could estimate the seizure activity using the measured EEG signal (Barja, 2024). This breakthrough had profound implications, allowing us to pinpoint and localize the source of seizures within the brain, as demonstrated in Figure 1 (Barja, 2021).

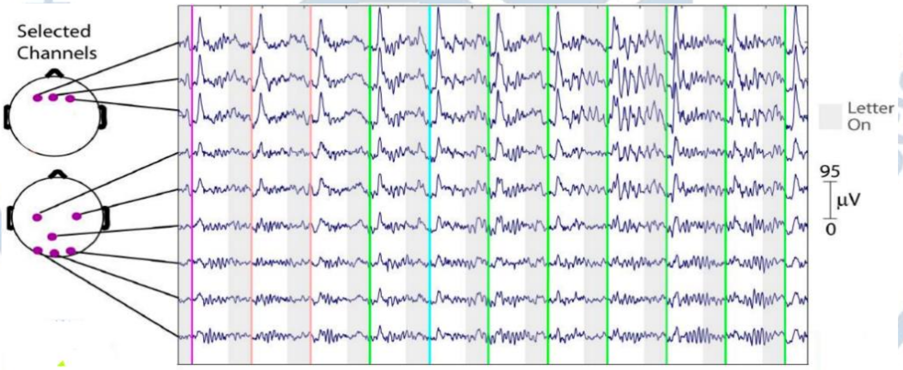

Figure1: Brain activity captured by EEG during an epileptic episode

Moreover, the prior integral equation (2.1) was implemented in MATLAB, yielding unambiguous results in the referenced study as shown in figure 2.

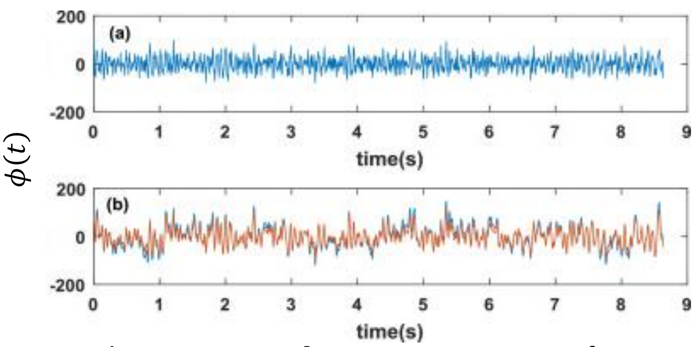

Figure2: Integral Equation Mapping of EEG Signals During a Seizure

2.1 Definition (Switzer, 2017): Let T be a topological space. The homology groups of T, denoted by $H_n(T)$, are defined as follows:

- Singular homology: For each non-negative integer n, $H_n(T)$ is the quotient group of the group of $n-chains$, $C_n(T)$, by the group of $n-boundaries$, $B_n(T)$.
- Chains: An $n-chains$ is a formal linear combination of $n-simplices$ in T.
- Boundaries: The boundary of an $n-simplex$ is an $(n-1)-chains$.

2.2 Definition (Bajardi, 2021): let T_1 and T_2 be topological spaces, and let $f: T_1 \longrightarrow T_2$ be a homeomorphism. A property P is said to be topologically invariant if T_1 has property P, then T_2 also has property P. In other words, a property is topologically invariant if it is preserved by continuous, bijective, and open (or closed) mappings.

2.3 Definition (Carlsson & Vejdemo-Johansson, 2021): Topological Data Analysis (TDA) is a mathematical framework that uses topological tools to analyze and understand data sets. It involves the following steps:

1. Simplification: The data is transformed into a simplicial complex, denoted by ψ.
2. Persistence: The topological features of ψ are analyzed using persistent homology, which measures the significance of these features over a range of scales.
3. Interpretation: The results of the persistence analysis are interpreted to gain insights into the underlying structure and patterns of the data.

2.4 Theorem (Barja, 2024): The integral equation $\phi(t)$ is a linear equation during an epileptic seizure.

2.5 Theorem (Barja, in press): The integral equation $\phi(t)$ for EEG signals during an epileptic seizure is a continuous function of the EEG signal $\upsilon(t)$ and the seizure function $\mu(\tau)$.

3. Methodology and Findings

During the seizure, EEG electrodes are attached to the patient's scalp, and the resulting average potential differences (APD) are recorded. To standardize the locations from which the APD were recorded, a system of electrode placement was introduced, called the International ten-twenty System, see figure3.

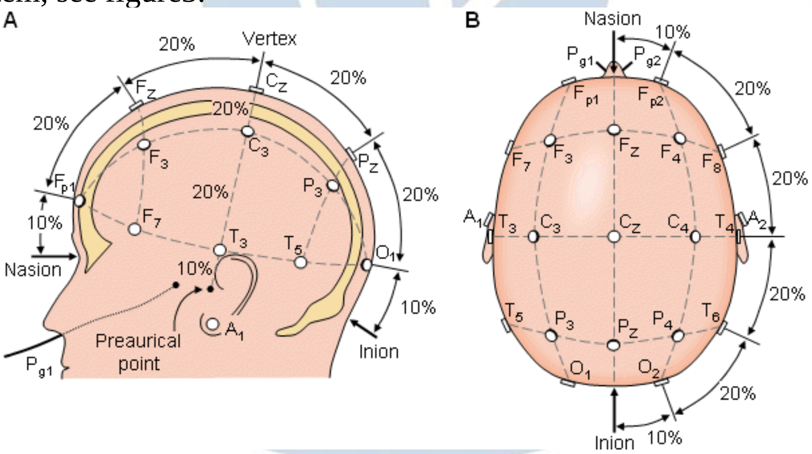

Figure 3: International 10-20 system

Before diving into the connection between the integral equation for EEG signals and visualizing topological patterns and dynamics, let's establish some essential assumptions that will guide our exploration.

1. Continuity of the Integral Equation: We assume that equation 2.1 represents a continuous mapping between the input functions $\upsilon(t-\tau), \mu(\tau)$ and the output function $\phi(t)$. This implies that small changes in the input function will lead to correspondingly

small changes in $\phi(t)$, ensuring that the essential features of the underlying dynamics are preserved.

2. Topological Invariance: Continuous mappings have the remarkable property of preserving topological properties under continuous deformations. This means that the topological structure of the reconstructed EEG signal, as represented by equation 2.1, is inherently determined by the topological properties of the kernel function $K(\tau)$ and the input functions $\upsilon(t-\tau), \mu(\tau)$.
3. Homology and Topological Features: Homology is a powerful tool that allows us to analyse the global structure of a space by studying its connected components, holes, and higher-dimensional features. In the context of the integral equation, homology can be used to relate critical points or singularities in $K(\tau)$ to the existence of specific topological features in the reconstructed EEG signal.
4. Topological Data Analysis: Topological data analysis (TDA) combines topology and data analysis to reveal the underlying structure of complex data sets. By applying TDA to the integral equation, we can analyse the topological features of $K(\tau)$,$\upsilon(t-\tau)$, and the reconstructed EEG signal, identifying features that are relevant for understanding the dynamics of the EEG signal during an epileptic seizure.
5. Applications of Topological Analysis: Insights gained from the topological analysis of the equation 2.1 have the potential to significantly improve the modelling and prediction of EEG signals. This, in turn, can lead to more accurate diagnosis and effective treatment strategies for epilepsy.

These essential assumptions provide a solid foundation for understanding the connection between the integral equation, topological concepts, and their applications in analysing EEG signals during epileptic seizures. The subsequent theorems presented in this work will provide a formal mathematical foundation for these assumptions, solidifying our understanding of this complex phenomenon.

3.1 Theorem: Let $K(\tau)$ and $\upsilon(t)$ be continuous functions defined on their respective domains. Then, the integral equation 2.1 represents a continuous mapping between the input function $\upsilon(t)$ and the output function $\phi(t)$, and this mapping is topologically invariant under continuous deformations of $K(\tau)$ and $\upsilon(t)$.

Proof.

Continuity of the Mapping: to establish the continuity of the mapping, we consider the integral operator defined by:

$$\int_t^{\infty} K(\tau)\, \upsilon(t-\tau)\mu(\tau)d\tau$$

Since $K(\tau), \upsilon(t-\tau)$, and $\mu(\tau)$ are continuous functions, when τ is fixed, $K(\tau)$ and $\mu(\tau)$ function as constants with respect to t. Therefore, the product $K(\tau)\mu(\tau)$ is also a constant with respect to t. Consequently, the expression $K(\tau)\, \upsilon(t-\tau)\mu(\tau)$ is a continuous function of t for each fixed τ, as it combines the constant factor $K(\tau)\mu(\tau)$ with the continuous function $\upsilon(t-\tau)$. As t changes, $\upsilon(t-\tau)$ varies continuously, resulting in small changes in the product $K(\tau)\, \upsilon(t-\tau)\mu(\tau)$. The integral of a continuous function over the interval $[t,\infty)$ is also continuous with respect to t. Therefore, the mapping $\phi(t)$ is continuous.

Topological Invariance: let $K'(\tau)$ and $\upsilon'(t)$ be continuous functions derived from $K(\tau)$ and $\upsilon(t)$, through continuous deformations. We define the new mapping as

$$\int_t^{\infty} K'(\tau)\, \upsilon'(t-\tau)\mu(\tau)d\tau \; = \phi'(t)$$

Since $K'(\tau)$,$\upsilon'(t-\tau)$, and $\mu(\tau)$ are continuous, the product $K'(\tau)\,\upsilon'(t-\tau)\mu(\tau)$ is also continuous for each fixed τ. As t varies, $\upsilon'(t-\tau)$ changes continuously, ensuring that $\phi'(t)$ is continuous as well. The mappings $\phi(t)$ and $\phi'(t)$ are topologically equivalent, as the continuous deformations imply that $K'(\tau) \longrightarrow K(\tau)$ and $\upsilon'(t) \longrightarrow \upsilon(t)$ maintain the same topological structure. Therefore, the essential topological features of the reconstructed signal $\phi(t)$ are preserved under these deformations, confirming the topological invariance of the mapping, as required.

3.2 Theorem: Let $K(\tau)$,$\upsilon(t-\tau)$, and $\mu(\tau)$ be continuous functions that possess well-defined homological properties. Then, the homology of the reconstructed integral equation of EEG signal during an epileptic seizure $\phi(t)$ can be described in terms of the homology of $K(\tau)$ and $\upsilon(t)$.

Proof.

Since the integral operator 2.1 can express this operator as a linear mapping:

$$\mathcal{L}(\,\upsilon(t)) = \phi(t), \text{where } \mathcal{L}(\,\eta) = \int_{t}^{\infty} K(\tau)\,\eta(t-\tau)\mu(\tau)d\tau$$

The operator $\mathcal{L}$ is continuous and linear since it is constructed from continuous functions (2.4 theorem). Specifically, the product $K(\tau)\,\upsilon(t-\tau)\mu(\tau)$ is continuous for all t and τ, ensuring that the integral $\phi(t)$ is well-defined and continuous. The homological properties of $\mathcal{L}$ can be analysed using properties of integral transformations, which provide insights into how the structure of the input functions influences the output.

To establish a relationship between the homology of $\phi(t)$ and the homologies of $K(\tau)$ and $\upsilon(t)$, we recognize that the homology of the output $\phi(t)$ is determined by the operator $\mathcal{L}$. Specifically, this relationship can be expressed in terms of homology groups:

$$H\big(\phi(t)\big) \cong H(\mathcal{L}) \cong H(K(\tau)) \oplus H(\upsilon(t))$$

This indicates that the homology of the reconstructed EEG signal $\phi(t)$ is fundamentally linked to the homological properties of both the kernel function $K(\tau)$ and the pre-seizure signal $\upsilon(t)$. Thus, we conclude that the homology of $\phi(t)$ can be described in terms of the homology of $K(\tau)$ and $\upsilon(t)$, as required.

3.3 Theorem: Let $K(\tau)$ and $\upsilon(t)$ be continuous functions representing the kernel and input functions in the integral equation 2.1. Then, topological data analysis (TDA) techniques can be utilized to explore the topological features of $K(\tau)$,$\upsilon(t)$ and the reconstructed integral equation $\phi(t)$. These analyses can yield valuable insights into the dynamics of the EEG signals during seizures.

Proof.

To apply topological data analysis (TDA) techniques to the integral equation 2.1, let us consider the functions $K(\tau)$ and $\upsilon(t)$ as continuous functions. This continuity allows us to analyse their topological properties through their persistent homology. We assume that a simplicial complex ψ can be constructed based on the data represented by $K(\tau)$ and $\upsilon(t)$. We begin by defining the vertices of ψ as points in the domain of $K(\tau)$ and $\upsilon(t)$. Let $\mathrm{T} = \{\big(t_i, K(\tau_j), \upsilon(t_k)\big)\}$ represent sampled values of K and υ. The edges of the simplicial complex are formed by connecting vertices based on a distance metric on figure 3, while higher-dimensional simplices are created by connecting edges according to proximity by the data distribution.

Once the simplicial complex ψ is constructed, we compute its homology groups $H_k(\psi)$ for various dimensions k. The $k-th$ homology group is defined as:

$$H_k(\psi) = \frac{ker(\delta_k)}{im(\delta_{k+1})}$$

Where δ_k is the boundary operator mapping $k - chains$ to $(k-1) - chains$. The interpretation of these groups provides insights into the topological features of the functions.

For instance, $H_0(\psi)$ indicates the number of connected components, $H_1(\psi)$ reveals cycles or holes, and $H_2(\psi)$ corresponds to voids or higher-dimensional holes. Therefore, by analysing the homology groups $H_k(\psi)$, we uncover essential topological features of $K(\tau)$ and $\upsilon(t)$. These features can be correlated with the dynamics of $\phi(t)$ equation, enhancing our understanding of seizure activity. As a result, the TDA of $\phi(t)$ can provide valuable insights into the dynamics of the EEG signal, as required.

4. Discussion

The three theorems presented in the previous section collectively enhance our understanding of the relationship between EEG signals during epileptic seizures and their topological properties. 3.1 Theorem establishes that the integral equation 2.1 represents a continuous mapping, ensuring that small changes in input functions $\upsilon(t)$ lead to proportionate changes in the output $\phi(t)$, which preserves essential seizure dynamics. This continuity reassures researchers and clinicians about the reliability of mathematical models in capturing brain activity. 3.2 Theorem correlates the homological properties of the reconstructed EEG signal with those of $\upsilon(t)$, suggesting that specific topological features can serve as indicators of seizure activity, thereby aiding in targeted interventions. Finally, 3.3 Theorem integrates topological data analysis (TDA) techniques, revealing significant patterns in EEG data that traditional methods might overlook. By analysing the homology groups of the constructed simplicial complex, this theorem provides insights into seizure dynamics that may lead to improved diagnostic tools and therapeutic strategies. Collectively, these theorems not only validate the application of persistent homology and TDA in neuroscience, but also pave the way for future investigations into the complex interplay between topological features and neurological phenomena, ultimately enhancing clinical decision-making in epilepsy management.

5. Conclusion

This study demonstrates the effective application of persistent homology in analysing EEG signals during epileptic seizures. By framing the EEG signals as integral equations, we established a robust mathematical framework that integrates topological data analysis with neurological research. The findings reveal significant topological features that correlate with the dynamics of seizure activity, providing deeper insights into the structure of brain function during epileptic episodes.

6. Future Directions

To further enhance the understanding and application of topological analysis in EEG research, several avenues for future studies can be explored:

- Real-Time Monitoring: Developing and integrating fuzzy topological data analysis methods for real-time EEG monitoring, addressing signal uncertainties and improving seizure prediction.
- Machine Learning Integration: Combining fuzzy topological data analysis methods with machine learning to enhance seizure classification by capturing subtle and ambiguous topological patterns.
- Diverse Data Analysis: Testing fuzzy topological data analysis methods on EEG data from diverse patient populations to improve generalizability in epilepsy diagnosis and management.
- Longitudinal Studies: Using fuzzy topological data analysis in longitudinal research to track topological feature evolution, offering insights into epilepsy progression and treatment effectiveness.

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التماثل المستمر في تحليل إشارات تخطيط كهربية الدماغ أثناء نوبات الصرع: تصور الأنماط والديناميكيات التبولوجية

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الملخص

يعد تخطيط كهربية الدماغ، المعروف باسم EEG، أداة قيمة ومفضلة لتشخيص اضطرابات الدماغ المختلفة، وخاصة الصرع، نظرًا لطبيعته غير الجراحية وقدرته على توفير رؤى شاملة لوظيفة الدماغ من خلال الجهد الكهربائي. ومع ذلك، غالبًا ما يُنظر إلى نتائج تقييمات تخطيط كهربية الدماغ أثناء النوبات الصرعية على أنها ضوضاء وليست نمطًا منظمًا. يمكن النظر إلى إشارات تخطيط كهربية الدماغ أثناء نوبة الصرع كمعادلة تكاملية، ويوفر الهيكل التبولوجي الجبري أداة لتحليل شكل وبنية بيانات إشارة تخطيط كهربية الدماغ من خلال تطبيق مفاهيم جديدة مثل التماثل والتماثل المستمر. في هذه الدراسة، سوف نوضح تطبيق التماثل المستمر في تحليل إشارات تخطيط كهربية الدماغ أثناء النوبات الصرعية من خلال إنشاء تمثيل تبولوجي وديناميكي منظم للمعادلة التكاملية لاشارات تخطيط كهربية الدماغ أثناء النوبة الصرعية.

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الكلمات المفتاحية

إشارات تخطيط كهربية الدماغ، معادلة تكاملية، التماثل، تحليل البيانات التبولوجية